Calculus 1
Final Exam – Solutions
November 1, 2024 (8:30–10:30)

university of groningen

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1) Apply L'Hospital's Rule to evaluate the limit $\lim_{x\to 0} \dfrac{\arcsin(2x) - 2\arcsin x}{x^3}$. Indicate the results (e.g. limit laws, continuity, differentiation rules) used in each step.

Solution. The limit L has an indeterminate form of type "$0/0$" since $\arcsin 0 = 0$ and $0^3 = 0$. Thus we can directly apply l'Hospital's Rule to get

$$L = \lim_{x\to 0} \frac{\arcsin(2x) - 2\arcsin x}{x^3} \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{(\arcsin(2x) - 2\arcsin x)'}{(x^3)'} = \lim_{x\to 0} \frac{\dfrac{2}{\sqrt{1-(2x)^2}} - \dfrac{2}{\sqrt{1-x^2}}}{3x^2}.$$

Above we used the Difference Rule, the Chain Rule, the Constant Multiple Rule, the Inverse Rule (as in $(\arcsin x)' = (1-x^2)^{-1/2}$), the Power Rule. The limit on the right-hand side is also of type "$0/0$". So we may apply l'Hospital's Rule again to obtain

$$\lim_{x\to 0} \frac{\dfrac{2}{\sqrt{1-(2x)^2}} - \dfrac{2}{\sqrt{1-x^2}}}{3x^2} \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{\left(\dfrac{2}{\sqrt{1-(2x)^2}} - \dfrac{2}{\sqrt{1-x^2}}\right)'}{(3x^2)'} = \lim_{x\to 0} \frac{\dfrac{8x}{(1-(2x)^2)^{3/2}} - \dfrac{2x}{(1-x^2)^{3/2}}}{6x}.$$

Here we used the Difference Rule, the Constant Multiple Rule, the Chain Rule, and the (Generalized) Power Rule. Finally, after cancelling the common factors of x in the numerator and denominator, the resulting limit can be evaluated by Direct Substitution. We find that

$$L = \lim_{x\to 0} \frac{\dfrac{8}{(1-(2x)^2)^{3/2}} - \dfrac{2}{(1-x^2)^{3/2}}}{6} = \frac{\dfrac{8}{(1-0^2)^{3/2}} - \dfrac{2}{(1-0^2)^{3/2}}}{6} = \frac{8-2}{6} = \frac{6}{6} = 1.$$

Therefore the limit in question is equal to 1.

2) Use Taylor Series to find a and b such that $\lim_{x\to 0} \left(\dfrac{\sin 4x}{x^3} + \dfrac{a}{x^2} + b\right) = 0$.

Solution. The Taylor series expansion of $\sin \alpha$ around $\alpha = 0$ is

$$\sin \alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \dots$$

By substituting $\alpha = 4x$ we obtain

$$\sin 4x = 4x - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} - \dots = 4x - \frac{32x^3}{3} + \frac{128x^5}{15} - \dots$$

Therefore

$$b + \frac{a}{x^2} + \frac{\sin 4x}{x^3} = b + \frac{a}{x^2} + \frac{4x}{x^3} - \frac{32x^3}{3x^3} + \frac{128x^5}{15x^3} - \dots = b + \frac{a}{x^2} + \frac{4}{x^2} - \frac{32}{3} + \frac{128x^2}{15} - \dots$$

Note that all terms having positive powers of x tend to zero as $x \to 0$. Hence the limit is determined by the constant term $b - 32/3$ and terms with x^{-2} in them. The former has itself as the limit, i.e. $\lim_{x\to 0}(b - 32/3) = b - 32/3$, whereas the latter, that is $(a+4)x^{-2}$ has an infinite limit as $x \to 0$ unless the coefficient $a + 4$ is zero. Therefore the original limit is zero if and only if

$$b - \frac{32}{3} = 0 \qquad \text{and} \qquad a + 4 = 0$$

or, equivalently,

$$a = -4 \qquad \text{and} \qquad b = \frac{32}{3}.$$

To conclude, we have $\lim_{x\to 0}\left(\frac{\sin 4x}{x^3} + \frac{a}{x^2} + b\right) = 0$ if and only if $a = -4$ and $b = 32/3$.

3) Use integration to find the area of the surface obtained by rotating the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ about the x-axis (a, b are positive constants).

Solution. We obtain the surface by rotating the graph of the function $f(x) = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$, $-a \le x \le a$ about the x-axis. The surface area is given by the definite integral $A = \int_{-a}^{a} 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$. Let us first compute the integrand $f(x)\sqrt{1+[f'(x)]^2}$. We find that

$$f'(x) = \frac{-bx/a^2}{\sqrt{1-(x/a)^2}} \;\Rightarrow\; [f'(x)]^2 = \frac{(b/a)^2(x/a)^2}{1-(x/a)^2} \;\Rightarrow\; 1 + [f'(x)]^2 = \frac{1 + [(b/a)^2 - 1](x^2)^2}{1-(x/a)^2}$$

$$\Rightarrow\; \sqrt{1+[f'(x)]^2} = \frac{\sqrt{1 + [(b/a)^2 - 1](x/a)^2}}{\sqrt{1-(x/a)^2}} \;\Rightarrow\; f(x)\sqrt{1+[f'(x)]^2} = b\sqrt{1 + [(b/a)^2 - 1](x/a)^2}.$$

Therefore the area is

$$A = 2\pi b\int_{-a}^{a} \sqrt{1 + [(b/a)^2 - 1](x/a)^2}\,dx.$$

Depending on the sign of $(b/a)^2 - 1$ this integral is one of three basic forms:

$$\int \sqrt{1-u^2}\,du, \qquad \int 1\,du, \qquad \int \sqrt{1+u^2}\,du.$$

Note that a and b are positive and therefore the three cases can be distinguished as follows:

$$(b/a)^2 - 1 < 0 \text{ iff } b < a, \qquad (b/a)^2 - 1 = 0 \text{ iff } b = a, \qquad (b/a)^2 - 1 > 0 \text{ iff } b > a.$$

If $b = a$, then $A = 2\pi\int_{-a}^{a} a\,dx = 4\pi a^2$ (as expected; in this case the surface is a sphere of radius a).

If $b < a$, then denote $\boxed{e := \sqrt{1 - \left(\frac{b}{a}\right)^2}}$ and substitute $u = e\frac{x}{a}$ to get $x = \frac{a}{e}u$ and $dx = \frac{a}{e}\,du$, and the limits of integration change to $-e$ and e. Therefore we obtain

$$A = \frac{2\pi ab}{e}\int_{-e}^{e} \sqrt{1-u^2}\,du.$$

The trigonometric substitution $u = \sin\theta$ ($\sqrt{1-u^2} = \cos\theta$, $du = \cos\theta\,d\theta$) lets us evaluate the integral

$$\int \sqrt{1-u^2}\,du = \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\theta + \sin\theta\cos\theta}{2} = \frac{\arcsin u + u\sqrt{1-u^2}}{2}$$

and find that

$$A = \frac{2\pi ab}{e}\left(\frac{b}{a}e + \arcsin e\right), \quad \text{where } e := \sqrt{1 - \left(\frac{b}{a}\right)^2}.$$

If $b > a$, then denote $\boxed{e := \sqrt{\left(\frac{b}{a}\right)^2 - 1}}$ substituting $u = e\frac{x}{a}$ yields $x = \frac{a}{e}u$ and $dx = \frac{a}{e}\,du$, and the limits of integration change to $-e$ and e. Therefore we obtain

$$A = \frac{2\pi ab}{e}\int_{-e}^{e} \sqrt{1+u^2}\,du.$$

The hyperbolic substitution $u = \sinh t$ ($\sqrt{1+u^2} = \cosh t$, $du = \cosh t\, dt$) lets us evaluate the integral:

$$\int \sqrt{1+u^2}\, du = \int \cosh^2 t\, dt = \int \frac{1+\cosh 2t}{2}\, dt = \frac{t}{2} + \frac{\sinh 2t}{4} = \frac{t + \sinh t \cosh t}{2} = \frac{\operatorname{arsinh} u + u\sqrt{1+u^2}}{2}$$

and therefore

$$A = \frac{2\pi ab}{e}\left(\frac{b}{a}e + \operatorname{arsinh} e\right), \quad \text{where } e := \sqrt{\left(\frac{b}{a}\right)^2 - 1}.$$

4) Evaluate the definite integral $\int_0^\pi \frac{1}{2+\cos x}\, dx$.

Solution. Let I denote the integral. The integrand is a rational function in $\sin x$ and $\cos x$ so we may simplify it by using the tangent half-angle substitution $u = \tan(x/2)$. This results in $\cos x = (1-u^2)/(1+u^2)$, $dx = 2/(1+u^2)$ and the limits of integration change accordingly: if $x = 0$, then $u = \tan(0) = 0$ and if $x \to \pi^-$, then $u = \lim_{x\to\pi^-} \tan(x/2) = \infty$. Therefore the substitution yields an improper integral

$$I = \int_0^\infty \frac{1}{2+\frac{1-u^2}{1+u^2}} \frac{2}{1+u^2}\, du = \int_0^\infty \frac{2}{2+2u^2+1-u^2}\, du = \int_0^\infty \frac{2}{3+u^2}\, du = \frac{2}{3}\int_0^\infty \frac{1}{1+\left(\frac{u}{\sqrt{3}}\right)^2}\, du.$$

Substituting $t = u/\sqrt{3}$ ($u = \sqrt{3}\,t$, $du = \sqrt{3}\, dt$) turns this integral into the basic inverse tangent integral (with the same lower and upper limits), that is

$$I = \frac{2}{3}\int_0^\infty \frac{1}{1+t^2}\sqrt{3}\, dt = \frac{2}{\sqrt{3}}\int_0^\infty \frac{1}{1+t^2}\, dt = \frac{2}{\sqrt{3}}\lim_{b\to\infty}\int_0^b \frac{1}{1+t^2}\, dt = \frac{2}{\sqrt{3}}\lim_{b\to\infty}[\arctan t]_{t=0}^{t=b}.$$

In the last step, we used the Fundamental Theorem of Calculus. Finally, we obtain

$$I = \frac{2}{\sqrt{3}}\left[\left(\lim_{b\to\infty}\arctan b\right) - \arctan 0\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - 0\right] = \frac{2}{\sqrt{3}}\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}.$$

The integral in question is therefore $I = \frac{\pi}{\sqrt{3}}$.

5) Solve the initial value problem $y'(x) + (\cos x)y(x) = 2xe^{-\sin x}$, $y(\pi) = 0$.

Solution. This is a first-order linear ODE and as such it can be solved using an integrating factor. The equation is of the form $y' + P(x)y = Q(x)$ with $P(x) = \cos x$ and $Q(x) = 2xe^{-\sin x}$. Therefore the integrating factor can be written as

$$I(x) = e^{\int P(x)\, dx} = e^{\int \cos x\, dx} = e^{\sin x}.$$

Multiplying both sides of the ODE by $I(x)$ yields

$$e^{\sin x}y' + e^{\sin x}(\cos x)y = 2x.$$

The left-hand side is the derivative of $e^{\sin x}y$ therefore we get

$$(e^{\sin x}y)' = 2x.$$

Integrating both sides with respect to x we obtain

$$e^{\sin x}y = \int 2x\, dx.$$

The integral on the right-hand side can be evaluated using the Power Rule. Thus we get

$$\int 2x\,dx = x^2 + C.$$

Therefore the general solution of the ODE $y' + (\cos x)y = 2xe^{-\sin x}$ is

$$y(x) = e^{-\sin x}(x^2 + C).$$

Setting $x = \pi$ yields

$$y(\pi) = e^{-\sin \pi}(\pi^2 + C) = e^0(\pi^2 + C) = \pi^2 + C$$

which, when compared to the initial value $y(\pi) = 0$, implies that $C = -\pi^2$. In summary, the solution of the initial value problem $y'(x) + (\cos x)y(x) = 2xe^{-\sin x}$, $y(\pi) = 0$ is

$$y(x) = e^{-\sin x}(x^2 - \pi^2).$$

6) Solve the following initial value problem

$$y''(x) + 2y'(x) + 5y(x) = 0, \quad y(0) = 0, \quad y'(0) = 2.$$

Solution. Looking for the solution in the form of an exponential function $y(x) = e^{rx}$, as we've done in class, we get the following auxiliary equation for the unknown coefficient r:

$$r^2 + 2r + 5 = 0.$$

The quadratic formula yields two complex roots:

$$r_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4\,(1)\,(5)}}{2\,(1)} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i.$$

Thus the two independent solutions are of the following form

$$y_1(x) = e^{r_1 x} = e^{(-1+2i)x} = e^{-x}e^{i(2x)}, \quad y_2(x) = e^{r_2 x} = e^{(-1-2i)x} = e^{-x}e^{-i(2x)}.$$

Using Euler's formula we can combine these solutions into a general (real function) solution. Namely, by taking

$$y(x) = A\frac{y_1(x) + y_2(x)}{2} + B\frac{y_1(x) - y_2(x)}{2i}$$

we get

$$y(x) = e^{-x}[A\cos 2x + B\sin 2x],$$

where A and B are *real* constants (to be fixed by the initial conditions). Now, the derivative of this solution is

$$y'(x) = e^{-x}[(2B - A)\cos 2x - (2A + B)\sin 2x].$$

For the initial conditions $y(0) = 0$, $y'(0) = 2$ to be satisfied, we must have

$$y(0) = e^0[A\cos 0 + B\sin 0] = A = 0,$$

and

$$y'(0) = e^0[(2B - A)\cos 0 - (2A + B)\sin 0] = 2B - A = 2.$$

Substituting $A = 0$ into the second condition yields $2B = 2$, implying that $B = 1$. Finally, plugging these parameter values into the general form yields the solution of the initial value problem:

$$y(x) = e^{-x}\sin 2x.$$